RTL may replace oscilloscope

An experimental study of the RLC resonance

Giacomo Torzo

Summer school AIF, Genova, July 27-31, 2007

1. Introduction

The resonant systems are an important part in physics teaching but rarely they are studied experimentally. In particular, the resonant RLC circuits are usually taught only theoretically because in the traditional laboratory one should use an expensive storage oscilloscope and complex data analysis in order to perform a complete quantitative study of the resonance.

On the other hand if one uses a real time data acquisition system (RTL), made of an interface with two or more analog inputs and of a Personal Computer, the experiment becomes much easier cheaper and faster, and moreover the experimental data may be stored for further analysis out of the laboratory (e.g. as home work).

One must only correctly choose the frequency range to be explored (i.e. the product LC) that should not be too high nor too small (a reasonable value for the resonant frequency is 1 kHz)

Here we describe the experiment performed using a widespread commercial RTL system (LabPro/ LoggerPro from Vernier) and a signal generator that provides an analog output signal V_f proportional to the frequency of the signal V_o.

We use a half-wave rectifier/filter (inside the dotted box in the example shown in figure 1) to measure the amplitude V_a of the signal across R or L or C.

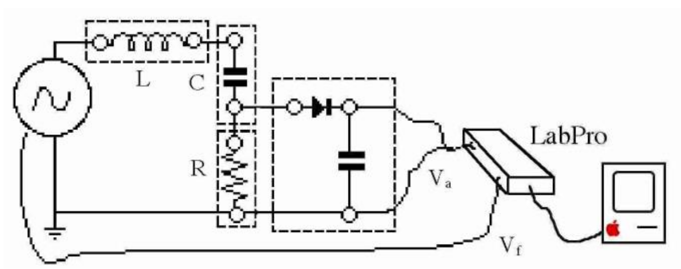

Figure 1: Circuit set up to measure the resonance across the resistance R

The circuit equation is

$$L\frac{dI}{dt}+\frac{\theta}{X}+PI=\varsigma_0\,\chi o\sigma\Box\tau$$

where V_o is the amplitude and ω is the angular frequency of the driving signal provided by the generator.

By substituting *I* = - *dq/dt* we obtain the differential equation for the charge *q(t)*:

$$\frac{d^2q}{dt^2}+\frac{P}{\Lambda}\frac{\delta\theta}{\delta\tau}+\frac{\theta}{\Lambda X}=\frac{\varsigma_0}{\Lambda}\chi o\sigma\Box\tau$$

whose stationary solution must be a signal with the same frequency as the driving signal.

The LabPro interface measures the values of V_a (voltage proportional to the amplitude of the oscillating voltage across the resistor) and V_f (voltage proportional to the frequency) while we sweep the frequency of the driving signal by acting on a potentiometer on the signal generator.

The measurement may be repeated after having placed the rectifying filter in parallel to the inductor L, or to the capacitor C.

Using the values R=1 KΩ, C=100 nF and L=300 mH, with a driving amplitude V_0=4.3V we obtained the graphs shows in figure 2, where the x-xis is still in voltage units.

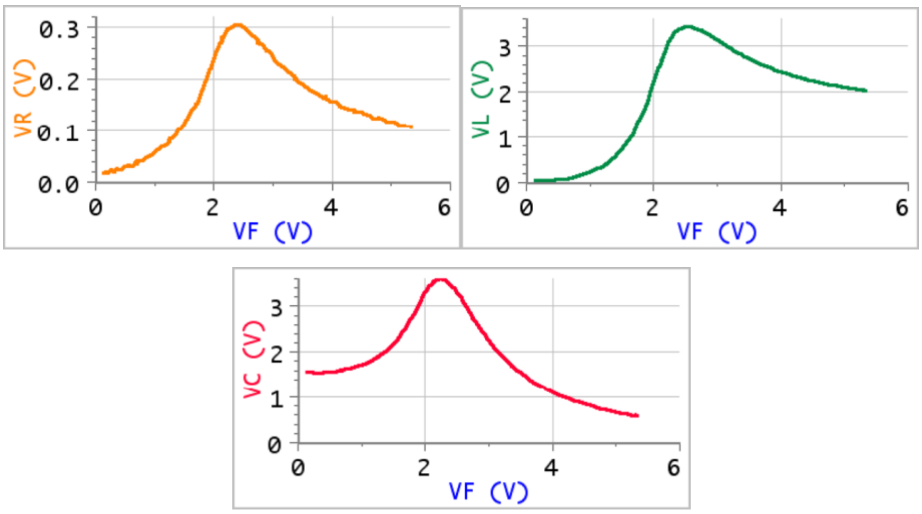

Figure 2: Voltage V_a measured across R, L and C, as a function of voltage V_f (frequency)

To calibrate the x-axis in frequency units, we may follow different procedures. If an oscilloscope or a frequency-meter is available we may follow the usual method of two points calibration, by typing in the two values of frequency measured independently.

On the other hand we may record the signal V_o (driving voltage) for two values of V_f (the voltage proportional to the frequency): then, using the cursor to read the period or the FFT utility in LoggerPro, we calculate the two frequency values corresponding to the two V_f values, and calculate the values (A=intercept B=slope) for the calibrating relation f=A+B* V_f

The graph of the voltage across R starts from zero at low frequency , and returns to zero at high frequency, passing through a peak value.

The graph of the voltage across C starts from the V_o values, increases to a peak value $V_p > V_o$ and decreases to zero at high frequency.

The graph of the voltage across L starts from zero at low frequency, increases to a peak value $V_p > V_o$ returning to V_o at high frequency.

We may note that the three peaks do not occur at the same frequency.

To understand these features of the RLC resonance we may exploit the graphing facilities of LoggerPro to build a model to be compared with the experimental results..

2. Model for the RLC resonance.

The Ohm's Law for circuits including inductors and capacitors must be written as

$$V(\omega) = Z(\omega)I(\omega),$$

where $Z(\omega)$ is the impedance of each element $(Z_R = R,\ Z_C = 1/j\omega C,\ Z_L = j\omega L)$

and $\omega = 2\pi f$ is the angular frequency[1].

The e.m.f. of the signal generator $V(t,\omega)$ is the sum of the voltage across the elements in series:

$$V(t,\square)=(P+\varphi\square\Lambda+\frac{1}{\varphi\square X})I(\tau,\square)=Z(\square)I(\tau,\square)$$

[1] For the inductor we must take into account the resistive component due to the dissipation on the iron core: in our inductor we found a resistive value at 1 kHz of about 100 ohm.

Writing the voltage signal as $V(t,\omega)=V_o exp(j\omega t)$ and the current signal as $I(t,\omega)=I_o exp(j(\omega t + \varphi)$, the total impedance $Z(\omega)=V(t,\omega)/I(t,\omega)=R+j\omega L+1/(j\omega C)$ may also be written $Z = Z_o exp(-j\varphi)$, and the signal across each element is obtained as partition ratio

$$|V_R/V_o| = |R/Z|; \quad |V_L/V_o| = |Z_L/Z| \quad e \; |V_C/V_o| = |Z_C/Z|$$

where $|A(j\omega)|$ is the module of the complex number $A(j\omega)$

Calculating the modules of Z_L Z_C and Z_o we get:

$|Z_L| = \omega L, \quad |Z_C| = 1/\omega C, \quad |Z| = |R+j\omega L(1-1/LC\omega^2)| = (R^2+L^2\omega^2(1-\omega_o{}^2/\omega^2))^{1/2}$

where we see that $\Box_0 = \sqrt{1/(\Lambda X)}$ is the value that gives the minimum for $Z(\omega)$ $(Z(\omega_o)=R)$.

Using these relations we may write the amplitudes of the voltages across R, C and L as:

$$|V_R| = \varsigma_0 \frac{P}{\sqrt{P^2 + (\Box\Lambda\Box 1/\Box X)^2}} \qquad |V_C| = \varsigma_0 \frac{1/\Box X}{\sqrt{P^2 + (\Box\Lambda\Box 1/\Box X)^2}}$$

$$|V_L| = \varsigma_0 \frac{\Box\Lambda}{\sqrt{P^2 + (\Box\Lambda\Box 1/\Box X)^2}}$$

3. Comparing the model with experimental data

The three functions V_R, V_L, V_C may be displayed (by defining them in the Menu Data / newColumn) in an overlapping graph: we will see that indeed they do not peak at the same frequency.

To understand how the peaks separation depends on the circuit parameters we must calculate the time derivatives of the three functions and find the zeros.

To simplify the notation we may let $\Box_0 = \sqrt{1/(LC)}$ and $\Box = R/2L$, so that we get $V_R/V_0 = 1/\sqrt{1+(\omega/2\gamma)^2[1-(\omega_0/\omega)^2]^2}$ and immediately we see that he maximum for V_R/V_o is for $\omega = \omega_o$.

For V_C we find the maximum at $\Box = (\Box_0^2 \Box 2\Box^2)^{1/2} \Box \Box_0 (1 \Box \Box^2/\Box_0^2)$

and for V_L the maximum is at $\omega = \omega_0^2/(\omega_0^2 - 2\gamma^2)^{1/2} \approx \omega_0(1+\gamma^2/\omega_0^2)$

The peak separation is therefore $\Delta\omega_{LC} = 2\gamma^2/\omega_0$: the greater is the damping γ the larger is the peak separation.

3. Measuring more resonance curves at the same time

Using two differential voltage probes and one voltage probe, we may collect three resonance curves in a single run. We must also use three rectifying filters as well (see figure 3).

Using LabPro we must know that the fouth analog channel does accept only voltages in the range 0+5V (not compatible with the frequency output Vf of our signal generator) Therefore our setup was (with reference to figure 3) V_f → CH1, $|V_C|$ → CH2, $|V_L|$ → CH3, $|V_R|$ → CH4, using a probe ±10V in must be measured by diferential voltage probes while $|V_C|$ may be measured using a standard ±10V probe.

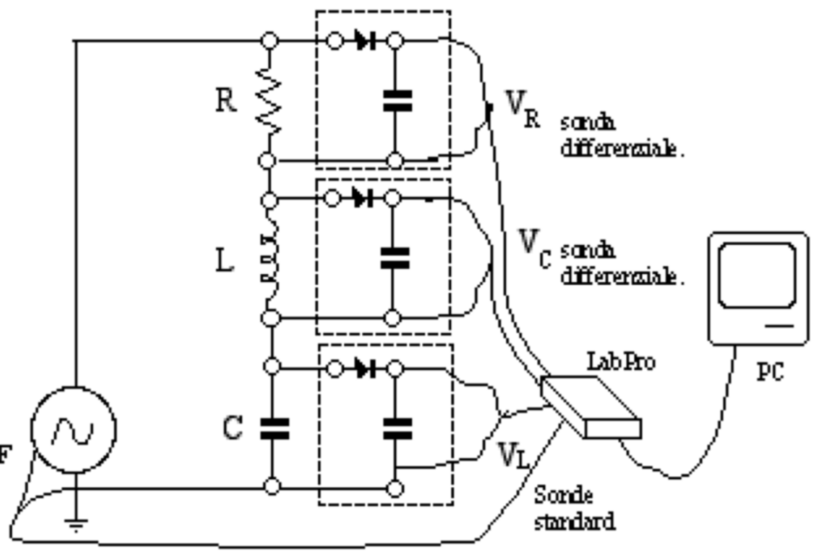

Figura 3: Circuit to measure simultaneously three resonance curves .

By repeating the measurements with a larger value of the resistor (R = 1000 Ω), we find that the peak separation is larger (see figure 4).

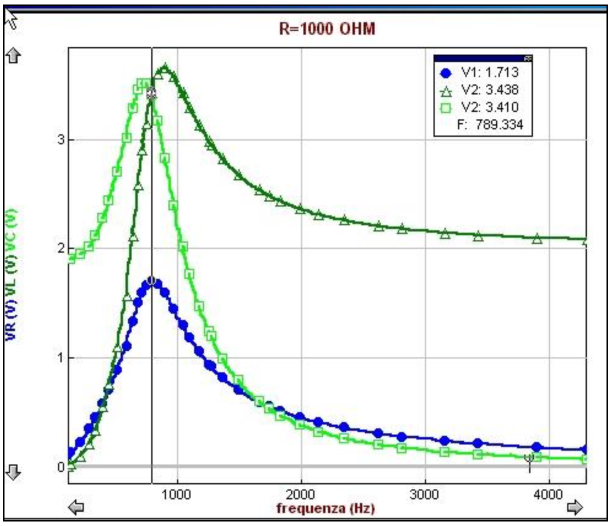

Figure 4: Resonance curves across R (closed circles) L (triangles) and C (open circles)

The peak separation between V_L e V_C predicted by the model with γ=1666 and ω_o=5773 rad/s, is $\Delta\omega_{\Delta X} = 2\gamma^2/\omega_0$ = 961 rad/s, and the experimental value is $\Delta\omega$ =954 rad/s.

4. The quality factor of the resonance

Our model predicts (at the resonant frequency) a decreasing value of the total impedance with R so that for R=0 the amplitude should diverge.

The width of the resonance peak (defined as the difference ω_2-ω_1 between the frequencies at which the amplitude reduces to $1/\sqrt{2}$ with respect to the peak value) is obtained by solving the following equation:

$$\frac{\varsigma_P}{\varsigma_0} = \frac{P}{Z} = \frac{1}{\sqrt{2}}$$

The two solutions are:

$$\omega_2 = \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} + \frac{R}{2L} = \sqrt{\gamma^2 + \omega_0^2} + \gamma \qquad \omega_1 = \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} - \frac{R}{2L} = \sqrt{\gamma^2 + \omega_0^2} - \gamma$$

We may easily see that $\omega_1 * \omega_2 = (\omega_o)^2$ i.e. $\omega_1 : \omega_o = \omega_o : \omega_1$; therefore ω_1 and ω_2 are not equally separated from ω_o, and the resonance curve is not symmetric.

The width is $\square\square = \square_2\ \square\square_1 = \frac{R}{L} = 2\square = \frac{\square_0}{Q}$ where $Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$ is named *quality factor* (*Q-value*) of the resonance.

While comparing the curves obtained with different values of the resistance R (e.g. R_1 = 1000 Ω, R_2 = 560 Ω , R_3 = 100 Ω) we must adjust the amplitude of the driving voltage: the smaller is R the higher are the peaks, and remember that the differential voltage probes have a range of about ±5V.

We may also note that the quality factor depends also on L and C: at constant R it increases with L and decreases with C.

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

We may change C and L to test this relation.