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RTL PENDULUM WITH CONTROLLED DAMPING AND TILTABLE OSCILLATION PLANE: A USEFUL TOOL FOR NEW LAB CURRICULA

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ABSTRACT
We propose a new tool for laboratory curricula based on computer-aided data acquisition and analysis. A pendulum coupled to a low-friction rotary sensor offers variable length, variable mass, variable oscillation plane (to change the effective gravitational restoring torque) and two different kind of damping torque: "dynamic friction" (almost constant) and "viscous" proportional to the angular velocity. Simple models implemented in common spreadsheets allow to compare the experimental results with the theoretical predictions.

KEYWORDS
Real Time Laboratory, numerical computation, pendulum, sliding friction, viscous friction, modeling

EXPERIMENTAL SETUP

Our pendulum is made of a perforated rubber ball (approximating a point-mass) attached to a thin aluminum rod (a knitting needle) whose end is fixed to the rotary sensor axis (figure 1). The pendulum effective length can be varied by sliding the rubber ball along the rod. The mass may be changed by using rubber balls of different sizes. The angular deviation from horizontal position of the pendulum's rotation axis can also be varied, and measured by a goniometer mounted close to the encoder.

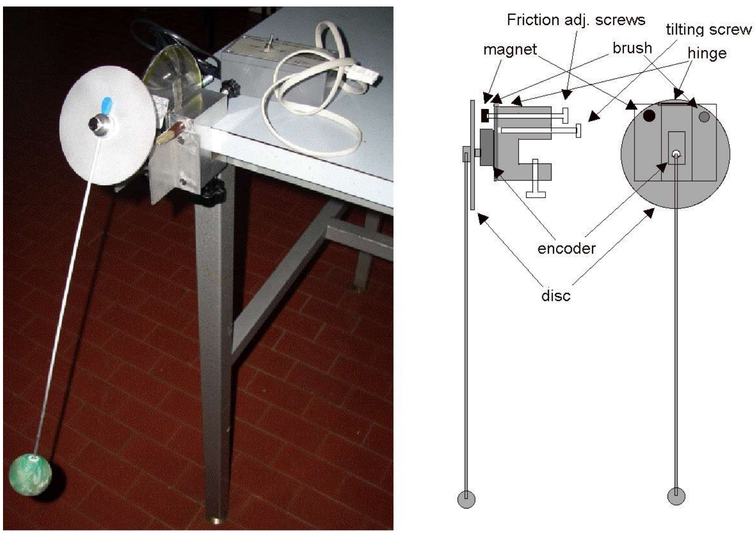

Figure 1. The RTL pendulum. Left: picture, right: schematics

When the rotation axis is horizontal, the oscillation may be damped by two kinds of resistant torque T_R: 1) a *viscous drag* provided by a magnet placed close to *an aluminum disc* fixed to the rotary sensor axis (the Foucault currents due to the magnet-disc interaction produce a torque proportional to the angular velocity of the disc), or 2) a nearly constant torque (*dry friction*) provided by a small brush sweeping the disc. The intensity of each resistant torque may be varied by adjusting the position (of either the magnet or the brush) with respect to the disc.

The RTL data acquisition system (RTL = acronym for **R**eal **T**ime **L**aboratory) exploits an home-made rotary sensor with low-friction optical encoder[1], connected to a TI graphing calculator through CBL interface (or to a PC through a LabPro interface).

THEORETICAL MODEL

Let us consider a pendulum consisting of a uniform thin rod of length l and mass m rotating about a fixed pivot located at one of its end. The rod is attached to a uniform disc of mass m' and radius r centered on the pivot and damped by a resistant torque T_R. A point–like mass M is fixed on the rod at the distance L from the pivot.

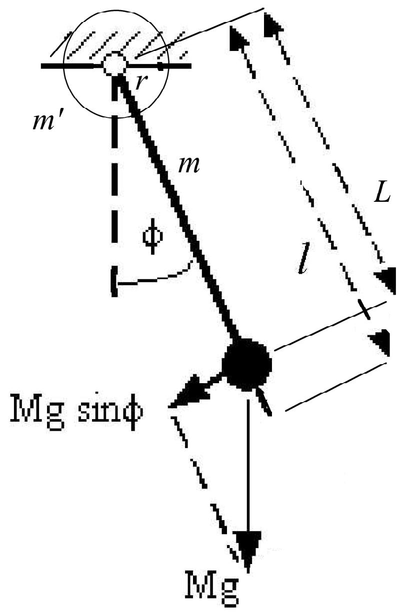

Figure 2. Sketch of forces for the ideal pendulum

The motion equation may be obtained by equating the time derivative of the angular momentum ($I\,d\omega/dt$) to the net torque $T=(ml/2 + M\,L)\,g \sin\phi + T_R(\omega)$,

$$I\,d^2\phi/dt^2 = (ml/2 + M\,L)\,g \sin\phi + T_R(\omega)$$

where ϕ is the elongation, $\omega = d\phi/dt$ is the angular velocity, $I = 1/3\,ml^2 + M\,L^2 + 2/3\,m'r^2$ is the momentum of inertia, and $T_R(\omega)$ is the resistant torque whose value depends on the kind of damping mechanism.

The equation may be simplified into the form:

$$d^2\phi/dt^2 = -\,g\,F/L \sin\phi + T_R(\omega)/I$$

where the coefficient F accounts for the pendulum non-ideality (due to the masses of the rod and of the disc):

$$F = \left(1 + \frac{1}{2}\frac{ml}{ML}\right) / \left(1 + \frac{1}{3}\frac{ml^2}{ML^2} + \frac{2}{3}\frac{m'r^2}{ML^2}\right)$$

[1] See LEPLA Project http://www.lepla.edu.pl; we successfully tried also a Vernier rotary sensor (model RMV-BTD), by adding the aluminum disc, magnet and brush.

For $l » L$, $r<<l$, and $m,m'<<M$ we get $F » 1$, and the equation becomes the equation for *ideal pendulum* with damping:

$$d^2\phi/dt^2 = - g/L \sin\phi + T_R(\omega)/I \qquad [1]$$

For small oscillation amplitudes it may be approximated (by letting $\sin f » f$) as:

$$d^2\phi/dt^2 = - g/L\ \phi + T_R(\omega)/I \qquad [2]$$

which, for T_R = 0 (no damping) has the well known (stationary) harmonic solution with angular velocity $\omega=\sqrt{(g/L)}$ and period $T=2\pi\sqrt{(L/g)}$.

PENDULUM MOTION WITH DAMPING

1) *Dry friction.*

Let us consider first the case of *dry friction damping* (Molina 2004), produced by a brush sweeping the disc.

Assuming a *constant* friction torque with negative sign with respect to the sign of the angular velocity $\omega=d\alpha/dt$, we get T_R = +C for ω >0, T_R = –C for ω <0 , and T_R = 0 for ω =0, i.e. T_R = C sgn(ω) , and the motion equation becomes:

$$d^2\phi/dt^2 = - g/L\ s\ \phi - (C/I)\ \mathrm{sgn}(d\ \phi/dt)$$

Let us analyze small oscillations starting from initial elongation ϕ_0 at rest (ω_0 =0).
During the first half-oscillation the pendulum will sweep the total angle $\phi_0 + \phi_1$, and it will reach a situation ($\phi=\phi_1$, ω =0) where we may calculate the energy balance.
The initial potential energy $MLg(1-\cos\phi_0)$ equals the new potential energy $MLg(1-\cos\phi_1)$ plus the energy lost due to the dry friction work $C(\phi_0 + \phi_1)$, or:

$$MLg(\cos\phi_1 - \cos\phi_0) = C(\phi_0 + \phi_1).$$

By using the Werner formula, we have:

$$(\cos\phi_1 - \cos\phi_0) = 2\sin[(\phi_0 + \phi_1)/2]\ \sin[(\phi_0 - \phi_1)/2]$$

and using again the approximation $\sin f » f$:

$$(\cos\phi_1 - \cos\phi_0) = (\phi_0 + \phi_1)(\phi_0 - \phi_1)/2$$

the energy balance equation gives for the elongation decrement $\Delta\phi=(\phi_0 - \phi_1)$ during the half-period:

$$\Delta\phi = 2C/MLg$$

The calculation may repeated for the second half-period, leading to the same result.

As a conclusion: the elongation, during each half-period, decreases by the constant value $\Delta\phi=2C/MLg$.
For small oscillations, also the period is constant, thus the amplitude must decrease linearly with time.

How many periods can be completed by the damped pendulum? The pendulum may move in the gravitational field, only if the gravitational torque is larger than the resistant torque: i.e. when

$$M\,L\,g\,\phi_i > C$$

An evaluation of the friction torque C may be derived by the previous formula: $C=MLg\Delta\phi/2$, where the decrement $\Delta\phi$ is expressed in rad.

2) *Viscous damping*

Let us now analyze the case of *viscous damping*, produced by the magnet-disc interaction (torque proportional to the angular velocity).
The motion equation is now:

$$d^2\phi/dt^2 = -\,g/L\ s\ \phi\ -\ (\gamma/I)d\,\phi/dt$$

or, with the positions $\delta=\gamma/2I$ and $\omega_o^2 = g/L$:

$$d^2\phi/dt^2 + 2\delta\ d\phi/dt + \omega_o^2\phi\ = 0$$

This equation, for small damping ($\delta << \omega_o$) has the solution:

$$\phi = \phi_o\ e^{-dt}\cos wt$$

where $w=\sqrt{w_0^2-d^2}$ » w_0 .

The elongation amplitude should therefore decrease exponentially, i.e. the decrement during each half-period should *decrease* proportionally to the amplitude itself.

EXPERIMENTAL RESULT FOR DRY FRICTION AND VISCOUS DAMPING

The rotary encoder allows recording the elongation ϕ versus time. From the measured values we may calculate and graph the absolute values of ϕ, and use them to build a new plot of the peak values of ϕ (amplitude) versus time.

1) *Dry friction.*

In figure 3 we report two records of the oscillations (obtained with a pendulum length $l = 0.5$ m, a mass $M = 30$ g) for two values of the friction torque .

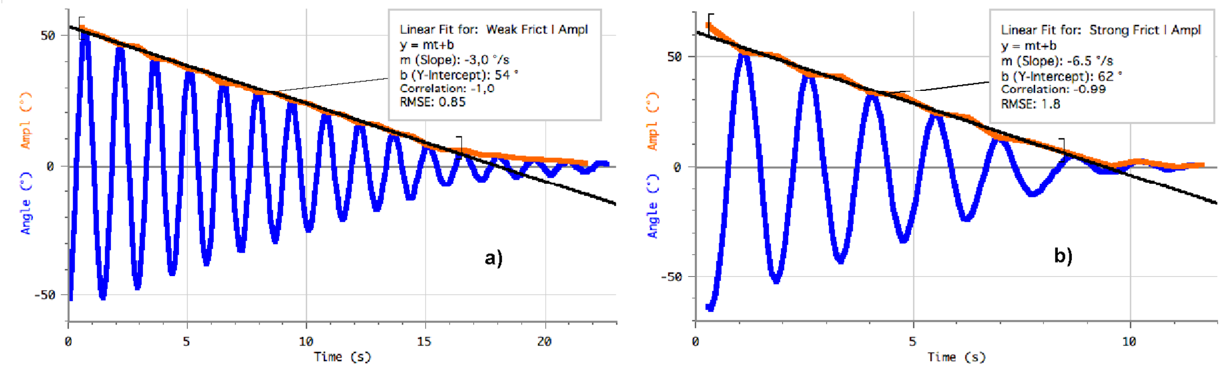

Figure 3: Oscillations recorded with different values of the dry friction

The expected linear dependence of the amplitude on time is apparent. The last oscillations in both graphs indicate that the damping decreases at very small elongations: this is due to the brush bristles elasticity that kills the sliding friction (when the disc motion is small, the bristles bend and their free end follows the disc rotation: the point contact is fixed on disc and no sliding occurs).

The values of the slope, in the linear fit of amplitude vs. time, are –3 degrees/s and –6.5 degrees/s, respectively. With a period given approximately by $T = 2\pi\sqrt{(L/g)} = 1.42$ s we may calculate in both cases the value of $\Delta\phi$, and we obtain for the friction torque $C = MLg\ \Delta\phi/2$ the two values $C = 5$ mJ and $C = 12$ mJ , respectively.

2) *Viscous damping*

The same procedure may be used to analyze the motion in case of viscous drag, due to the magnet-disc interaction.

In figure 4 we report the graphs of elongation vs. time and exponential best fits to the curves of amplitude vs. time, for two values of magnet-disc distance (1.5 mm and 1 mm , respectively).

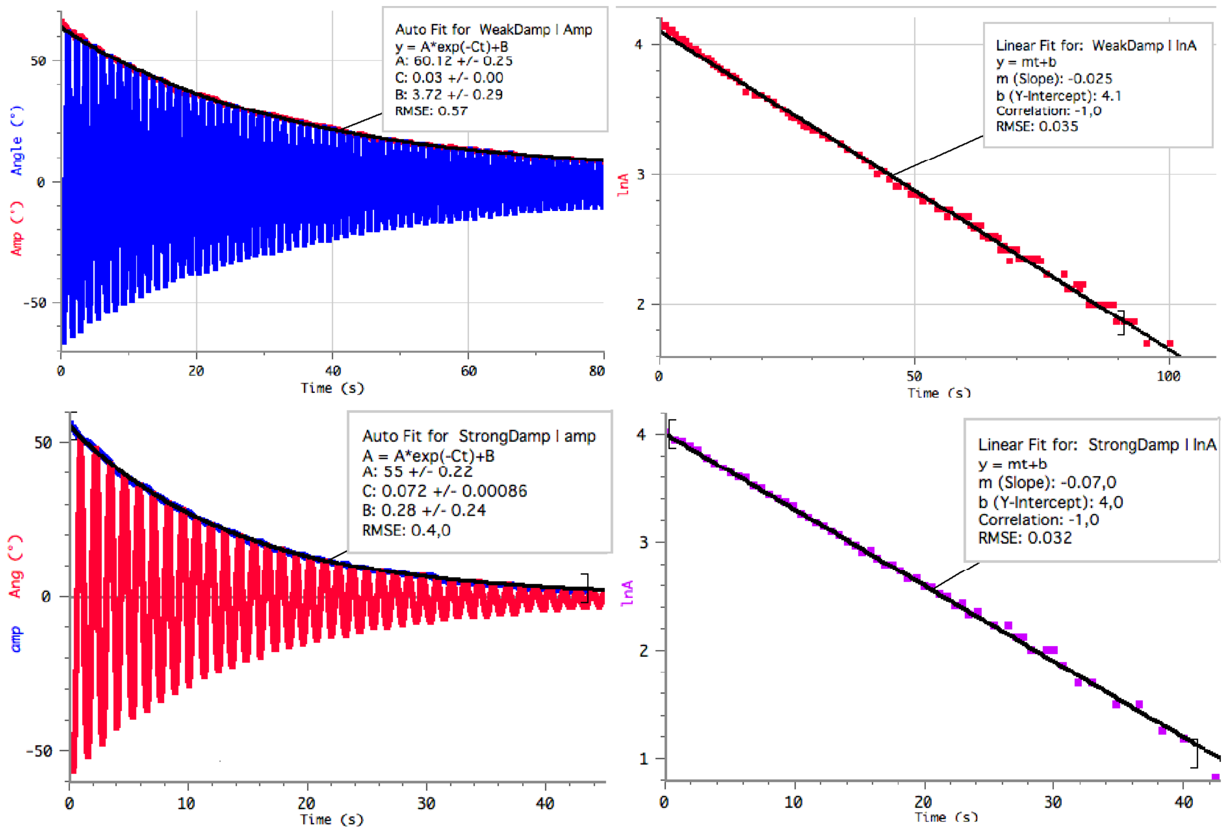

Figure 4: Oscillations recorded with different values of the viscous damping: the amplitude vs. time is fitted by an exponential function at left and a linear regression of the natural log of amplitude vs. time is shown at right.

The fitting curve is $f(t)=f_0 \exp(-dt)+f_1$, where the damping coefficient d is the reciprocal of the decay constant τ. Reducing the magnet-disc gap decreases the time constant from 33 seconds to 14 seconds.
Equivalent plots on the right side of figure 4 show the logarithm of amplitude decreasing linearly with time.

PENDULUM MOTION AT LARGE AMPLITUDE

The motion equation [1], valid for large amplitudes, cannot be easily solved by high-school students (and often not even by college students) using mathematical analysis, due to the difficulty introduced by the non linear term sinϕ in the restoring force. However computer simulation and iterative numerical computation may be used to overcome this difficulty.

The software may be any spreadsheet (as Excel), and the numerical method may be that proposed by R. Feynman (1968) in his famous *Lectures on physics* to study planetary motions.

The Feynman method

By studying the pendulum motion at discrete time intervals of arbitrary length Δt we may define the angular velocity $\omega(t+\Delta t)$ at the time $(t+\Delta t)$ in terms of the of mean acceleration α_m and the velocity $\omega(t)$ at the time t :

$$w(t+Dt)=w(t)+\alpha_m Dt$$

Similarly, the angular displacement $\phi(t+\Delta t)$ at the time $(t+\Delta t)$ may be written in terms of the average angular velocity ω_m:

$$f(t+Dt)=f(t)+w_m Dt$$

These equations give us the values of angle and angular velocity after the time interval Δt if we know $\phi(0)$, $\omega(0)$ and $\alpha(0)$. By iteration we may calculate these values for $t = 2\,\Delta t$, $t = 3\,\Delta t$, and so on. For average velocity Feynman uses the velocity calculated at the time in the middle of the interval, and for the first midstep assumes $\omega(\Delta t) = \omega(0) + (\Delta t/2)\alpha(0)$.

Let us first consider the case of *no damping*. From the tangential acceleration $a = g\sin\phi$ (calculated as $a = f/m$ from the tangential force $f = mg\sin\phi$) we obtain the angular acceleration $\alpha = (g/l)\sin f$, with ϕ expressed in radians.

With the Feynman method we should use in the spreadsheet the formulas:

$\alpha(t) = -(g/l)\sin\phi(t)$

$\omega(t+\Delta t/2) = \omega(t-\Delta t/2) + \alpha(t)\,\Delta t$

$\phi(t+\Delta t) = \phi(t) + \omega(t+\Delta t/2)\,\Delta t$

An equivalent approach consists in using the *initial* acceleration in computing the velocity and the *final* velocity in computing the angle, using in the spreadsheet the simpler set:

$\alpha(t) = -(g/l)\sin\phi(t)$

$\omega(t+\Delta t) = \omega(t) + \alpha(t)\,\Delta t$

$\phi(t+\Delta t) = \phi(t) + \omega(t+\Delta t)\,\Delta t$

Figure 5 shows the result of the iterative calculation for not damped pendulum, with $l = 0.5$ *m*, $\Delta t = 0.05$ s and $\phi(0) = 90°$ (large amplitude oscillations)

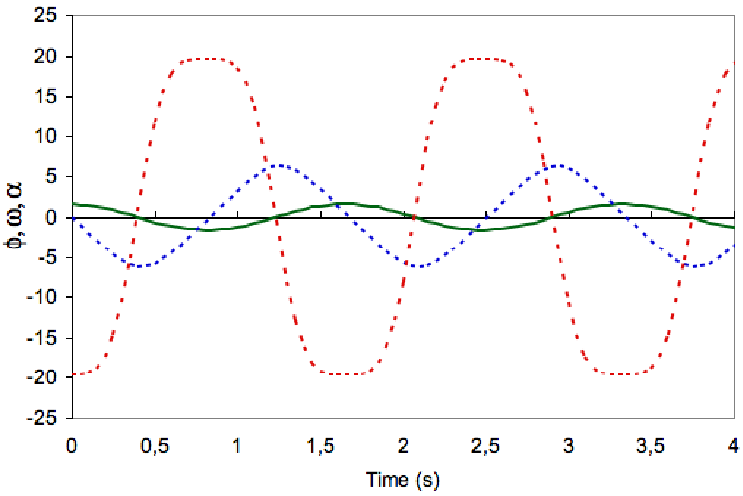

Figure 5: Calculated elongation ϕ (line), angular velocity ω (dotted line), angular acceleration α (dashed line) vs. time

We may observe that the predicted motion is *not* harmonic (while the angle plot $\phi(t)$ might appear as a sinewave, the acceleration plot $\alpha(t)$ is clearly *not sinusoidal*).

Prediction for damped pendulum

If we introduce into the model a damping force (a force that is always *opposite* to the velocity) a corresponding acceleration must be inserted into the motion equation:

In the case of sliding "dry friction" we have:

$$\alpha(t) = -(g/l)\cdot\sin f(t) - C\cdot sign[w(t)]$$

where C is a constant.

In the case of viscous damping

$$\alpha(t) = -(g/l)\cdot\sin f(t) - 2d\cdot w(t)$$

where 2 d is also a constant.

Figure 6 shows the prediction for sliding friction with *l = 0.5 m*, Δt = 0.03 s, ϕ (0) = 90° and C = 0.5 rad/s^2. We may observe that the behavior is quite similar to that shown in figure 3a.

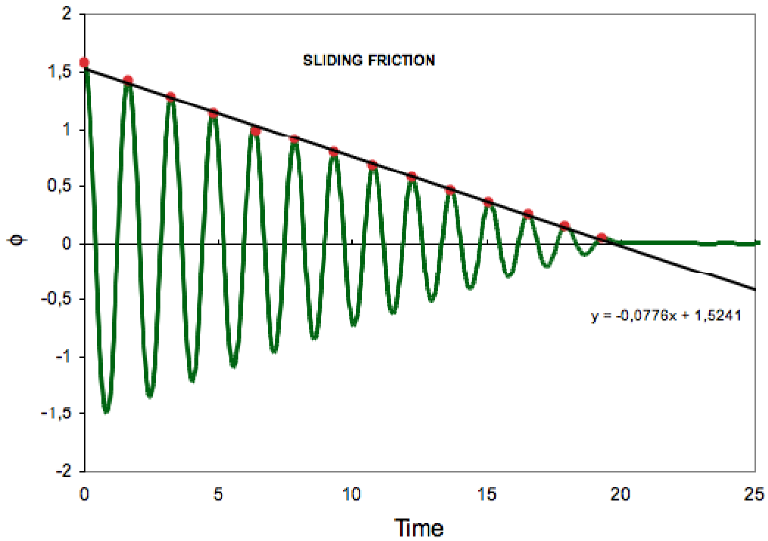

Figure 6: Simulated oscillations with dry friction

Figure 7 shows the prediction for viscous friction with *l = 0.5* m, Δt = 0.03 s, ϕ (0) = 90° and 2δ =0.3 s^{-1}.

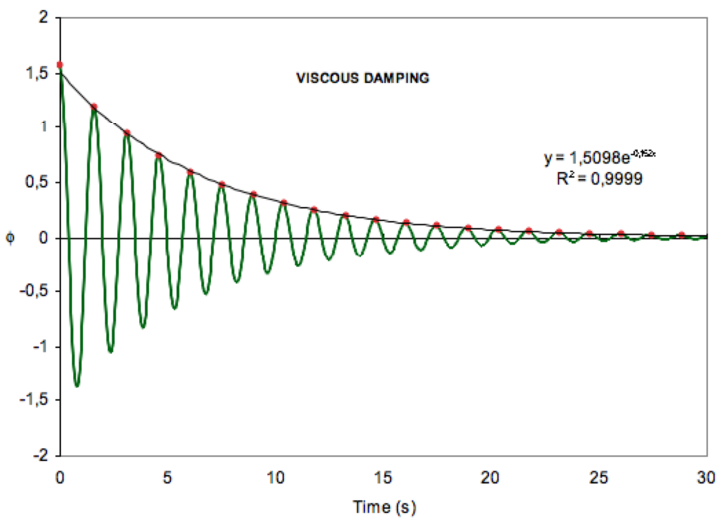

Figure 7: Simulated oscillations with viscous damping

COMPARING SIMULATION AND EXPERIMENTAL DATA

By properly adjusting the parameters (initial values, friction coefficients) in the numerical calculations, we may test our models on the experimental results, as shown in figure 8 and 9.

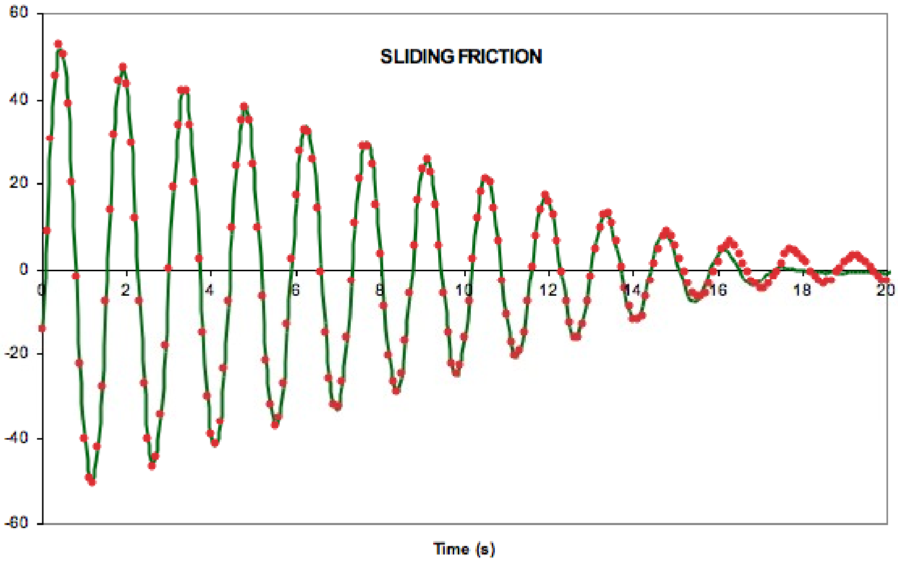

Figure 8: The line represent the values calculated with the "sliding friction" model using C = 0.35 rad/s^2, and the dots are the measured values.

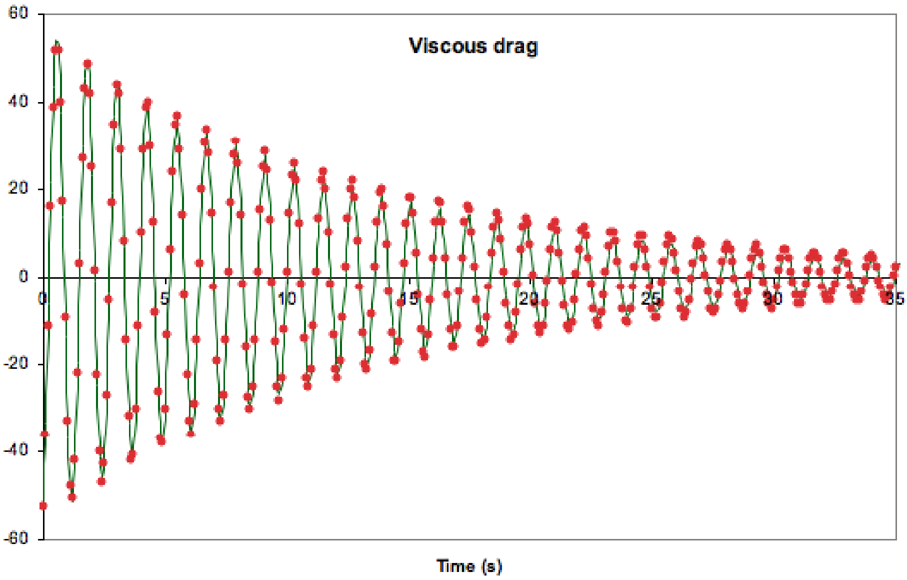

Figure 9: The line represents the values calculated with the "viscous friction" model using $2\delta = 0.153\ s^{-1}$, and the dots are the measured values

THE PENDULUM MOTION WHEN THE ROTATION AXIS IS TILTED WITH RESPECT TO THE HORIZONTAL DIRECTION.

When the pendulum rotation axis is tilted by an angle β, with respect to the usual horizontal position, the component *g'* of the gravitational force acting on the mass *M* which produces the driving torque is reduced by the factor cos β.
Therefore the harmonic solution of the motion equation will predict a reduced angular velocity $\omega' = \sqrt{(g'/L)} = \sqrt{(g\cos\beta/L)}$ and an increased period $T' = 2\pi\sqrt{(L/g\cos\beta)}$.

Table 1: The predicted and measured values of the period for different tilt angles

Angle *β* (degrees)	$(\cos\beta)^{1/2}$	*T* '(s)	T_{meas} (s)
0	0.000	1.41	1.41
5	0.997	1.41	1,40
10	0.99	1,42	1,41
15	0.98	1,44	1,42
20	0.96	1.47	1.44
25	0.95	1.48	1.45
30	0.93	1.51	1.48
35	0.90	1.56	1.52
40	0.87	1.62	1.57
45	0.84	1.68	1.61
50	0.80	1.76	1.68
55	0.75	1.88	1.79
60	0.70	2.01	1.85

In Table 1 we report a comparison between the predicted and measured values for the period for different values of the tilt angle β.

The same data are graphically shown in figure 10 :

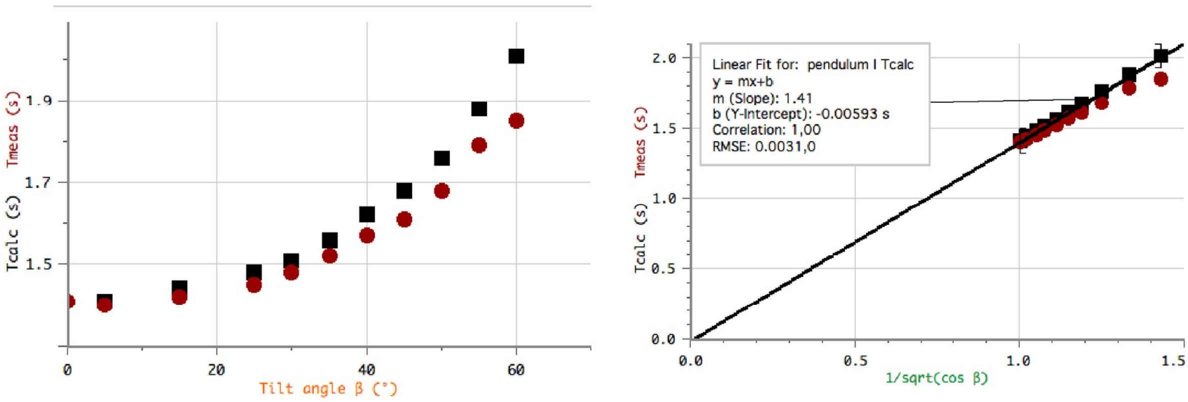

Figure 10: Calculated (squares) and measured (circles) periods vs. the tilt angle β and vs. $(\cos\beta)^{-1/2}$

The expected dependence of the measured period on the inverse square root of cos β is substantially confirmed.

The slight systematic difference between the measured periods and the values predicted by the model may be explained by considering that the thin rod is not perfectly rigid: it bends more and more when increasing the tilt angle (due to the mass weight) and this produces an increasing error in the measured tilt angle .

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