XXX International Physics Olympiad
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Workshop

Harmonic and anharmonic oscillations studied with CBL and Graphic Calculators

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Mechanic oscillating systems on-line

Using *on-line apparatuses* (sensors + interface + calculator) to study oscillatory motions may be very useful in the physics laboratory because the system *evolution in time* becomes directly accessible.

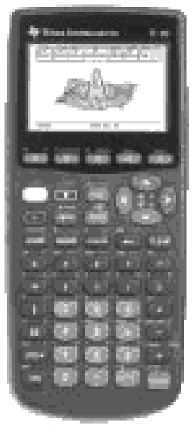
Plots of position, velocity and acceleration vs. time allow an immediate evaluation of the essential characteristics of the observed motion.

Also the experimental investigation within less common graphic representations (velocity/position, acceleration/position, force/acceleration, force/position ...) may offer precious hints for a deeper understanding of the studied phenomena.

To illustrate the potentialities the on-line experiments, we will show several oscillating systems: some of them are normally part of introductory courses in mechanics, while others are less commonly performed because they are strongly anharmonic.

A "pocket-size" on-line apparatus

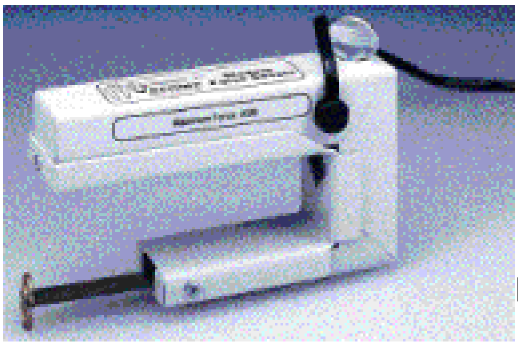
TI92



TI89

CBL (interface)

CBR



Force sensor

TEXAS INSTRUMENTS

Vernier SOFTWARE

T^3 :
Teachers
Teaching with
Technology

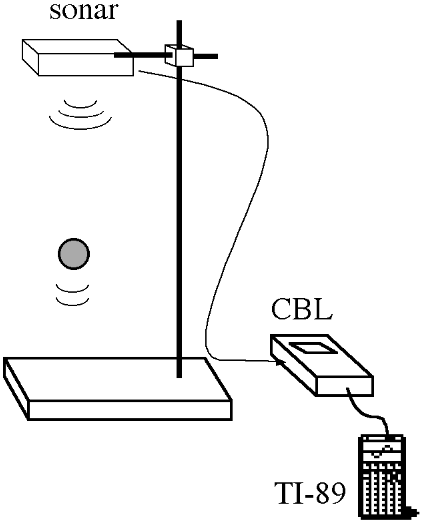
Using this simple & cheap on-line apparatus, many experiments on oscillations can be easily performed : Mass-spring oscillator, Pendulum, Atwood oscillator, Bouncing ball, Cart on incline, Galileo oscillator, Yo-yo (or Maxwell wheel), See-Saw on round or square pivot…

The majority of these phenomena are *intrinsically anharmonic* motions, as most of the ordinary-life mechanical oscillations. This is due to the fact that usually the restoring force is produced by some component of the *gravity field*, which is constant.

In some cases a quasi-harmonic motion may be obtained by choosing a proper system *geometry* (pendulum) and *small departures from equilibrium* position.

Intrinsically harmonic motion is produced only by a restoring force which is *linearly dependent on the displacement*, as in the case of mass-spring oscillator, or of the Atwood oscillator (when the immersed body has uniform cross section).

Bouncing ball

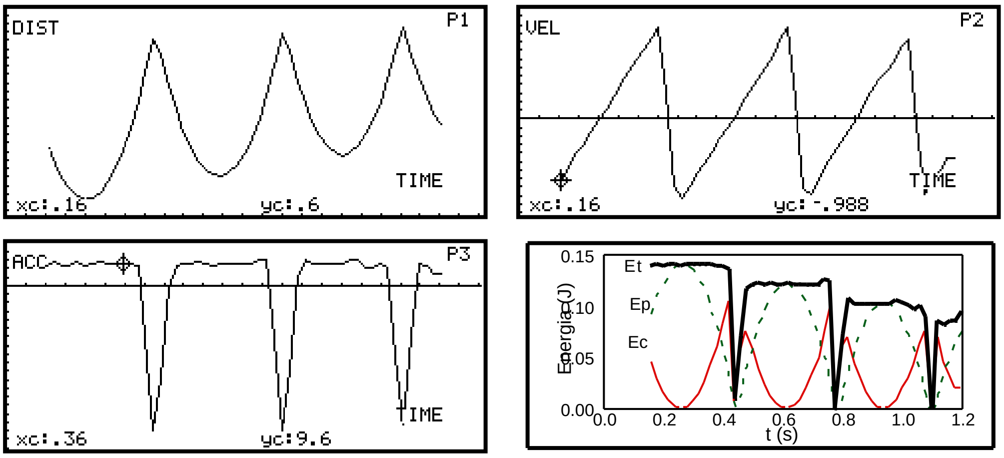


An ultrasonic motion detector (CBR), acting as a sonar, measures the distance of a sound-reflecting object by recording the time delay τ of the echo. The distance is computed as the product $x = \tau\, v$, where v is the sound speed in air.

At each sampling time also the velocity and the acceleration of the target body are calculated and recorded.

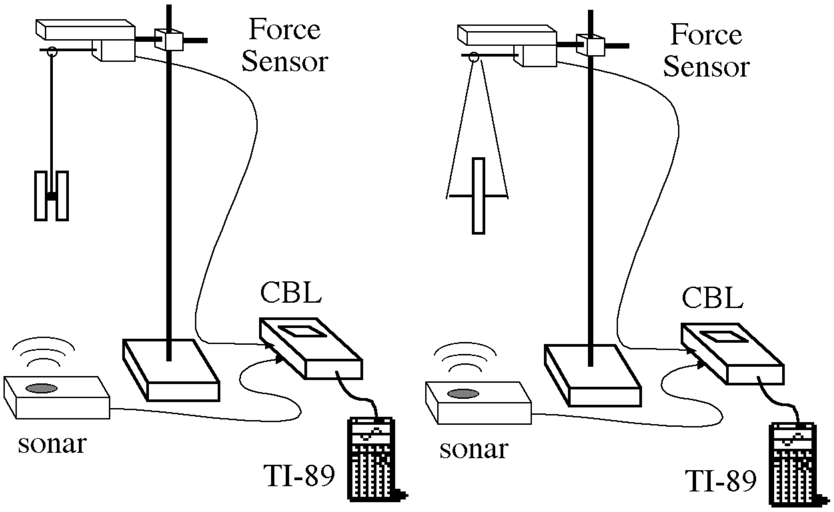
A ping-pong ball is left in free fall below the CBR (face-down); therefore the x-axis is vertical and directed downward, and the "restoring" force (gravity) is constant and positive.

The quasi-elastic properties of the ball and of the ground provide large negative acceleration pulses at each bounce.

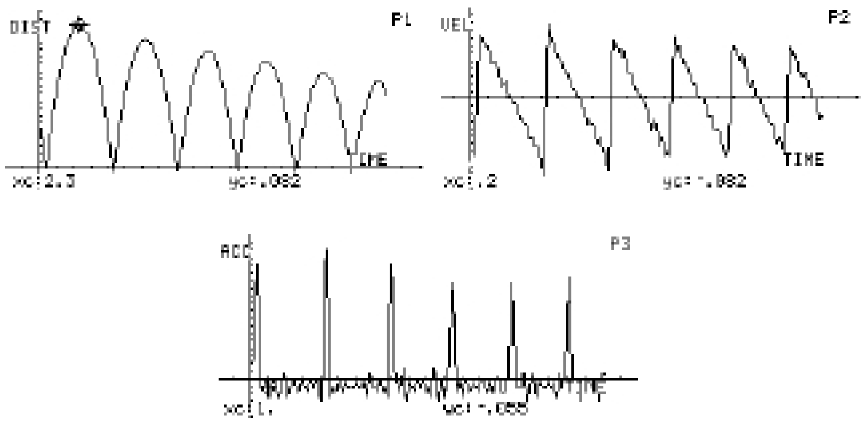


1. The plots x(t), v(t), a(t) vs. time give a complete description of the motion: x(t) is made of a series of parabolas, v(t) is made of a series of straight segments, a(t) is a constant (positive) value with large (negative) peaks corresponding to collisions with the ground.
2. By fitting v(t), within a single segment, a value of the acceleration can be obtained that is slightly less than 9.81 m/s^2.
3. Distance values x(t) may be transformed into height values $h(t)=x_o-x(t)$, upward measured from the origin at distance x_o. In this reference frame it is easier to calculate the gravitational energy U=mgh and the kinetic energy $Ec=(1/2)mv^2$.
4. The total energy ($E=U+E_c$) behavior may be studied in a plot vs. time: it clearly shows that energy is lost mainly during collisions, indicating some details of the collision process. The kinetic energy is temporary transformed into elastic energy (not detected by sonar), and only partially given back at the end of the collision.
5. Period T measurements for various rebounces show that oscillation is anharmonic. A plot of T vs. $\sqrt{X_{max}}$, fitted by a straight line, may be understood within the simple model of uniformly accelerated motion, where $4\sqrt{(2/a)}$ is the value of the predicted slope.

Maxwell Wheel (yo-yo)



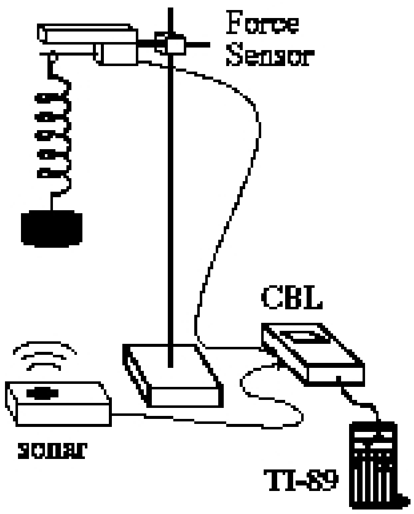
A yo-yo, or a Maxwell wheel, hanging from a fixed point, is placed above a CBR, facing upward.
The motion of the target is therefore measured in a reference frame with vertical x-axis directed upward.
The "restoring" force along x is therefore constant and negative while the torque (due to gravity and to wire tension) changes sign when the body reaches the wire-end.
If the wires are attached to a force sensor, this may detect the sharp peaks in the wire tension corresponding to acceleration peaks at the equilibrium position.



1. The plots x(t), v(t), a(t), f(t) vs. time are quite similar to those for a bouncing ball (with reversed x-axis).
2. The main difference is in that here rotation drastically reduces the linear acceleration: the angular velocity may be calculated from the relation between v and ω. In the energy balance the term corresponding to rotational kinetic energy must be taken into account.

See: B. Pecori and G. Torzo: "The Maxwell wheel investigated with MBL" The Physics Teacher, 36, 362-366 (1998)

Vertical mass-spring oscillator



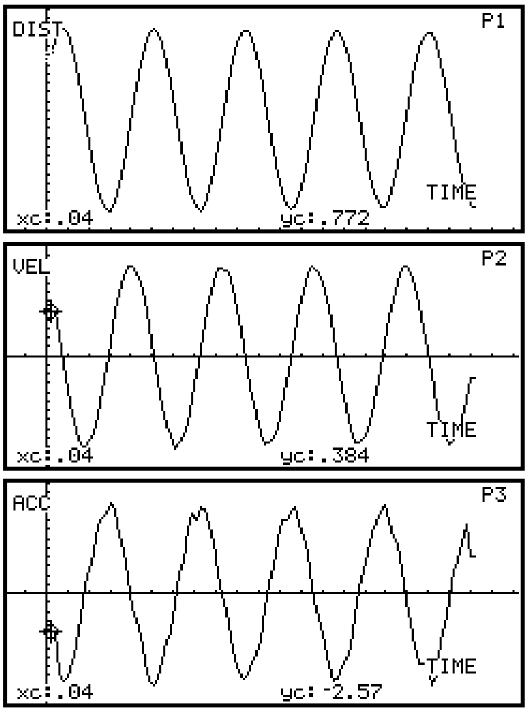
Elastic Force proportional to displacement from equilibrium position

$$\mathbf{F = - k\,x}$$

$$\mathbf{a = - (k/m)\,x = -\omega^2\,x}$$

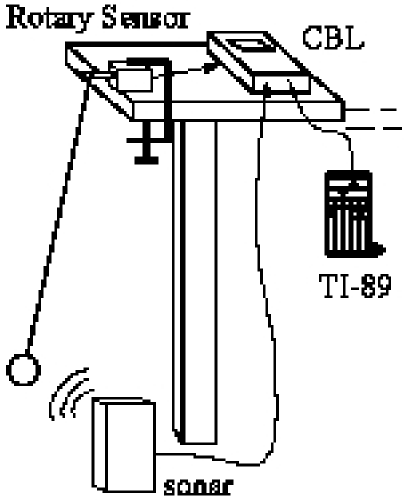
$$\mathbf{T = 2\pi\sqrt{m/k} = \textit{constant}\ (isochronism)}$$

(k= elastic constant , ω = angular frequency, T= period)



1. The plots x(t), v(t), a(t), f(t) position, velocity acceleration and force vs. time show a behavior clearly sinusoidal. It may be shown how to create fitting functions to various curves, phaseshifts may be discussed …
2. By moving the cursor on the screen, period measurements may be performed showing that the period is amplitude independent, as expected for an harmonic oscillator (isochronism)
3. The plot f(x) force-displacement results to be linear, and the slope obtained from the fit gives the elastic constant k of the spring.
4. The plot a(x) acceleration- displacement results to be linear, and the slope (k/m) obtained from the fit gives the inertial mass m_i, once known k.
5. The experiment may be repeated with a different mass and then with a different spring to show the period dependence on $\sqrt{m/k}$
6. Within the DataMatrix Editor environment, columns with the calculated values of potential energy (gravitational, elastic), kinetic energy and total energy may be generated. Plots of the calculated energies vs. time allow to demonstrate the energy transformations as well as to calculate the damping coefficient (nearly exponential decay). Oscillations at large amplitudes show that energy is dissipated discontinuously (larger dissipation corresponding to velocity peaks) by viscous friction.

Pendulum



The restoring torque is due to gravity and to pivot reaction.
Torque depends on the oscillation amplitude α

$$F = - m g \sin\alpha$$

$$M = - L m g \sin\alpha \qquad I = m L^2$$

(point mass approximation: L≈ pivot to center of mass distance)

$$a_{ang} = M/I = - (g/L) \sin\alpha$$

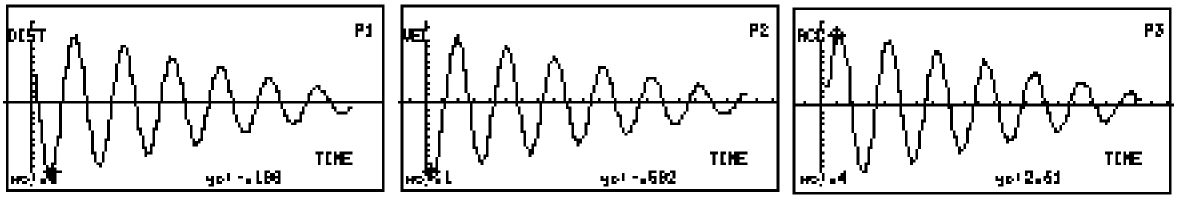
Torque is *not* proportional to angular displacement !

Only for small oscillations the following *approximation* is valid:

$$a_{ang} = - (g/L)\,\alpha = - \omega^2 \alpha$$

$$T = 2\pi\sqrt{L/g} = \textit{constant} \quad \text{(isochronism)}$$

(ω = angular frequency, T= period)



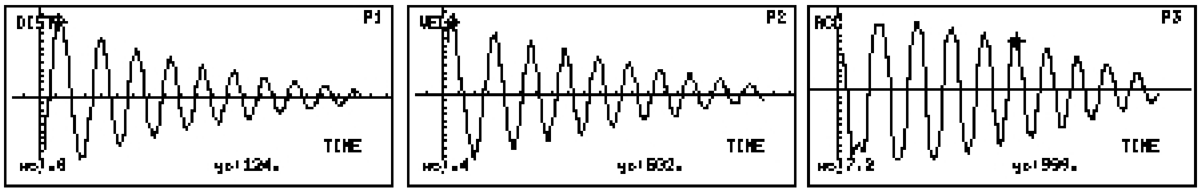
Small amplitude oscillations

The small amplitude oscillation may be studied using data taken with sonar. The plots x(t), v(t), a(t) appear sinusoidal.

Measurements of the period T vs. amplitude α indicate that the motion is isochronous.

It may be shown that T does not depend on the value of the mass while it does depend on its distance L from the center of oscillation.

The value predicted by the model ($T=2\pi\sqrt{L/g}$) can be compared with the one measured from the plots x(t), v(t), a(t).

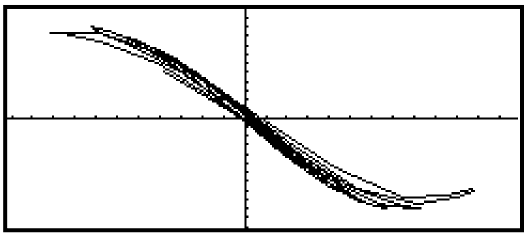


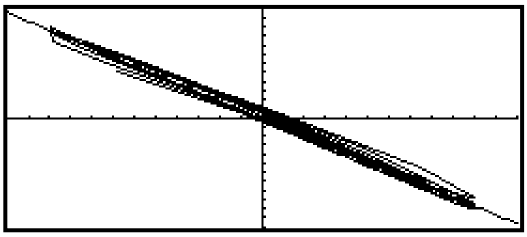
Large amplitude oscillations

The large amplitude oscillation may be studied using data taken with the potentiometric sensor (after converting voltages into angles α.

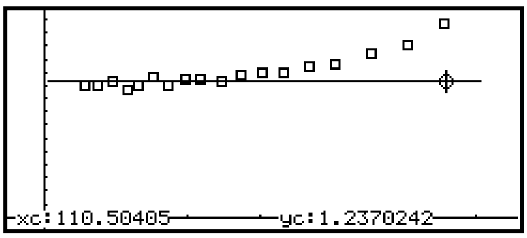
Here the distortion of the plots of $v_{ang}(t)$ and of $a_{ang}(t)$ with respect to damped sinusoidal funtions is clearly evident. Also the plot of a_{ang} vs. α is no more linear.

Instead a_{ang} is linear with sinα (as predicted by the model $a_{ang} = -g/L\sin\alpha$.





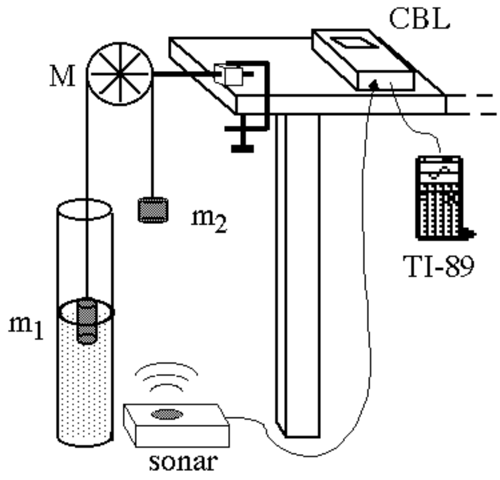
Acceleration vs. α   Acceleration vs. sin(α)



Period vs. amplitude

Also a plot of the period T vs. α will show that the pendulum is far to be isochronous.

Atwood Oscillator



in air:

Gravity Force

F= Δmg = *constant*

a = $\Delta mg/(m_1 + m_2 + M)$

($\Delta m = m_2 - m_1$, M = effective mass of the pulley)

in water:

Gravity Force and Archimede's Force

F = $\Delta mg - \rho Vg$ = *constant*

a = $(\Delta mg - \rho Vg) /(m_1+m_2+M)$

(ρ= liquid density, V= immersed volume)

if equilibrium is achieved for half-immersed cylinder **($\Delta mg = \rho V_{tot} g/2$)**:

$\alpha = (\Delta mg - \rho V_{tot} g) /(m_1 + m_2 + M) = -\Delta mg/(m_1+m_2+M)$

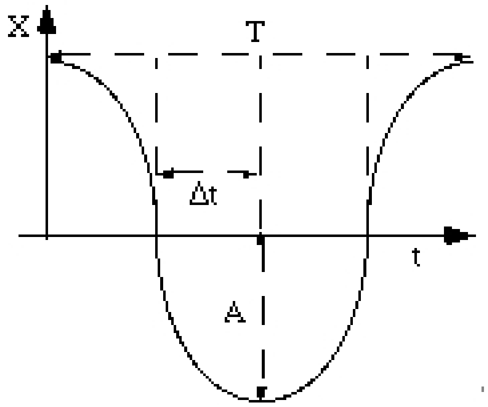
in air-water

Gravity Force, and Archimede's Force depending on immersed volume $V(x) = \pi r^2 x$

F = $\Delta mg - \rho V(x) g$

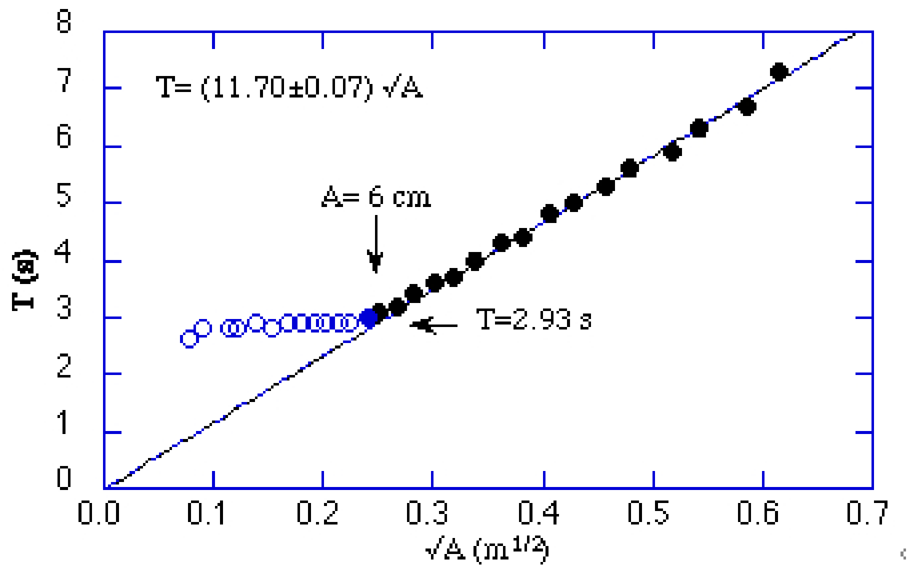
if equilibrium is achieved for half-immersed cylinder (L/2), $\Delta mg = (1/2)\rho g \pi r^2 L$

F = $-\pi \rho g r^2 x$ (*proportional to displacement)*



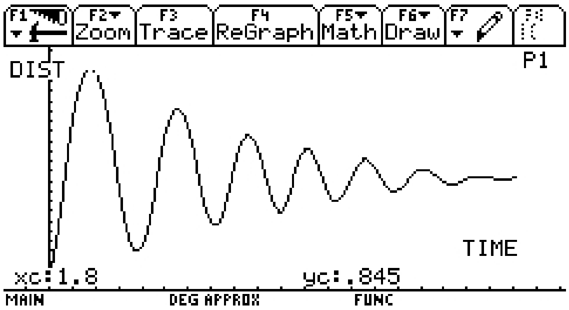
x(t) plot made of a series of parabolas corresponding to “fall” from height A upward (in water) and downward (in air).

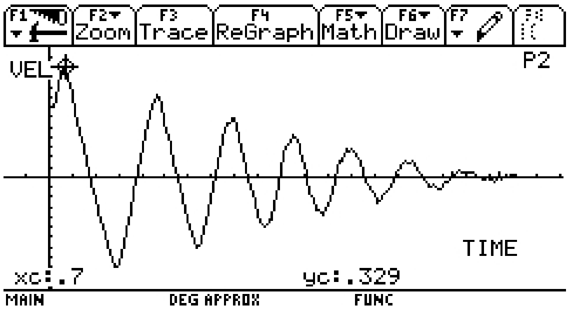
Predicted period : $T = 4\sqrt{2/a}\sqrt{A} \approx 12\sqrt{A}$



Transition from *anharmonic* to *harmonic* regime for decreasing amplitude

Atwood

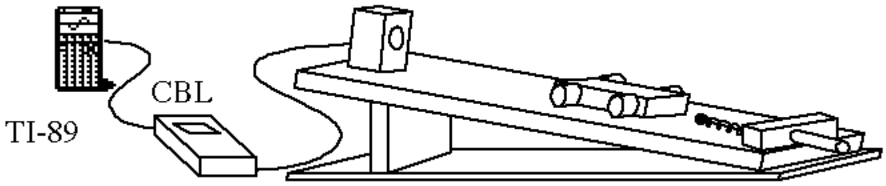


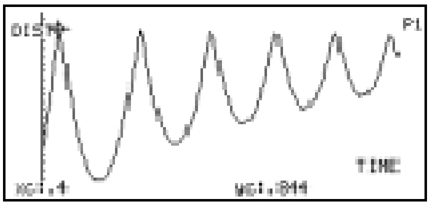


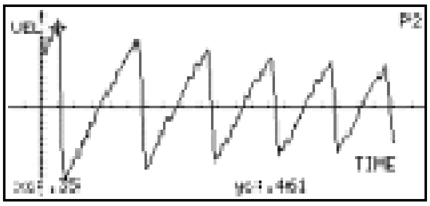
1. The plot x(t), vs. time looks like the corresponding one for the mass-spring oscillation, and this could suggest an harmonic behavior.
2. However the plots v(t) and a(t) demonstrate that this motion resembles that of a bouncing ball, of the cart on the incline and yo-yo: it is a uniformly accelerated motion during most part of the time. It can be studied following the same analysis.
3. In the case of small oscillations oscillations (with the cylinder always partially immersed) the motion turns into a damped harmonic motion .
4. By performing a full set of measurements of the period T and by studying the plot of T vs. √A, the transition from anharmonic to harmonic motion becomes evident.
5. An extended analysis of the time-evolution of the total energy, including dissipative effects may also be carried out

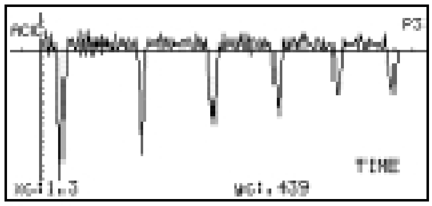
See: B.Pecori, G.Torzo, A.Sconza "Harmonic and Anharmonic Oscillations investigated by using a Microcomputer Based Atwood's Machine" *Am. J. Phys*, **67**, 228-235 (1999)

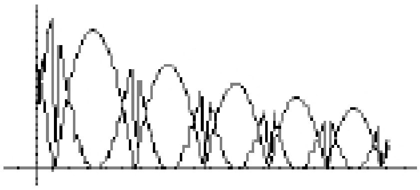
Cart rolling on incline



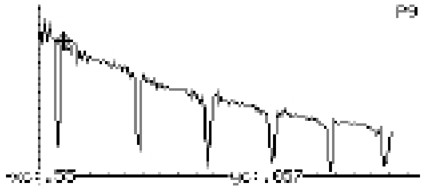








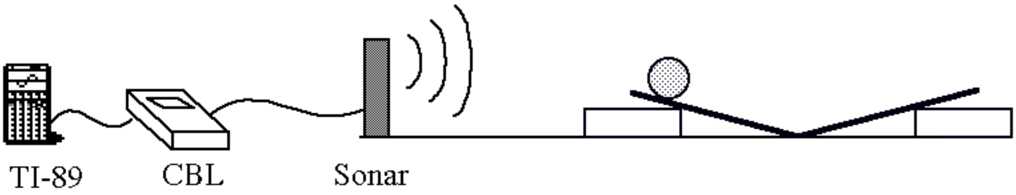
Ec, Ep vs. time



E=Ec+Ep vs. time

1. The plots x(t), v(t), a(t), f(t) vs. time are again similar to those for a bouncing ball, with the difference that here the v(t) plot shows slightly different slopes for upward and downward motions. From each fit the two values of the acceleration may be derived, and from their difference the friction coefficient may be calculated.
2. More acquisitions with different friction may be performed, and the energy ($mgx\sin\alpha+mv^2/2$) versus time plot shows that the energy loss is due both to the collisions and to the work done against friction.
3. The oscillation period may be measured and plotted versus $\sqrt{X_{max}}$, (the predicted slope is still $4\sqrt{(2/a)}$, but now the acceleration is $a=g\sin\alpha$).
4. By repeating the experiment with different values of the cart mass m and of the tilt angle α it may be shown that the period depends on α and not on m.

Galileo Oscillator



Restoring force (horizontal): component of the gravity force depending only on the inclination (±γ)

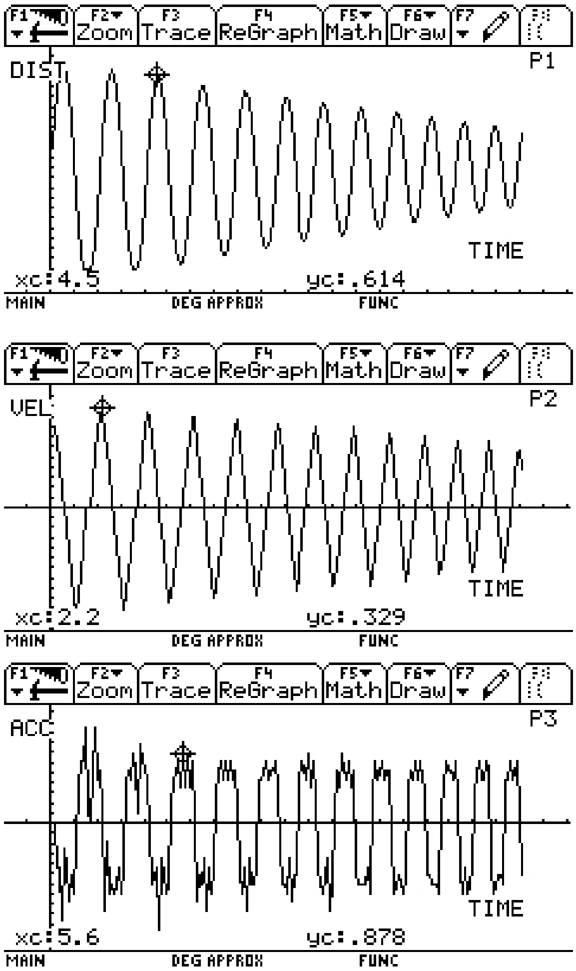
$$\mathbf{F_x = \pm m\ g\ sin}\ \gamma = \pm \mathit{constant}$$

$$\mathbf{a = \pm g\ sin}\ \gamma$$

The observed "horizontal oscillation" is due to the horizontal component of the gravity force acting on the center of mass of the sphere which is bound to move on the rail .
The vertical component of the speed changes direction each time the sphere goes through the lowest quote. In the vertical direction the center of mass behaves as a "bouncing ball".

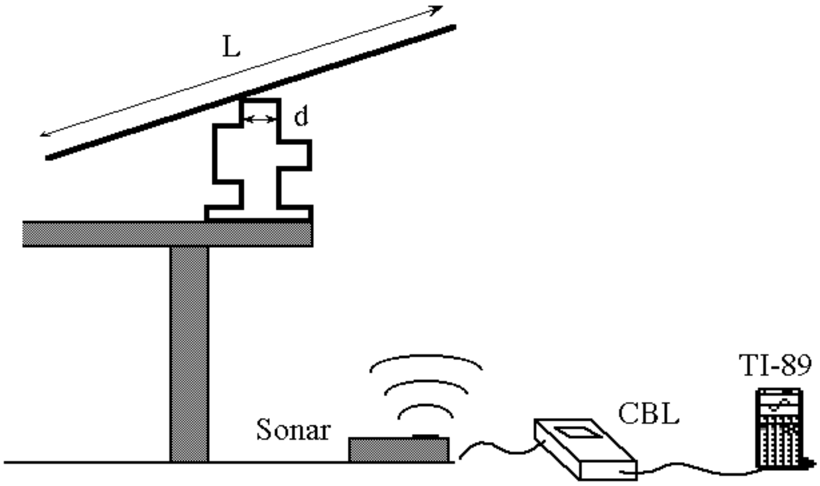
The horizontal component of the motion is isomorph to the Atwood oscillator: also here the restoring force "changes direction" when the body crosses the equilibrium position.

Oscillation is therefore *anharmonic*



1. The plots x(t), v(t), a(t), f(t) vs. time are similar to those for Atwood oscillator. Motion is indeed uniformly accelerated.
2. The restoring force is F= -mg sinφ sgn(x) , and therefore it depends on the tilt φ.
3. The measured acceleration (slopes in v(t) or mean values on a(t) plots) may be compared with the value predicted by the model: it will come out that the sphere moment of inertia must be taken into account. Using a ping-pong ball (instead of an omogeneous sphere) demonstrates the role of mass distribution in rotational dynamics.
4. The agreement between theoretical prediction and experimental values may be not satisfactory. By repeating the experiment with the same sphere and a larger rail the disagreement increases, (as well as using the same rail and a smaller sphere). The teacher may ask the students to try explaining the reason why (effect of a giration radius different from sphere radius).

See-Saw with double pivot



The restoring torque is due to gravity and to pivot reaction.
Torque depends on the oscillation amplitude φ because the effective component of the gravity force is *not constant*:

$$\mathbf{F = m\ g\ cos\ \phi}$$

$$\mathbf{M = (d/2)\ m\ g\ cos\ \phi \qquad I \approx mL^2/12}$$

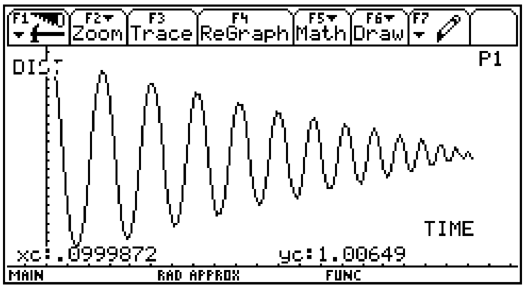
The lever arm of the torque is constant, and for small oscillations (and d<<L)

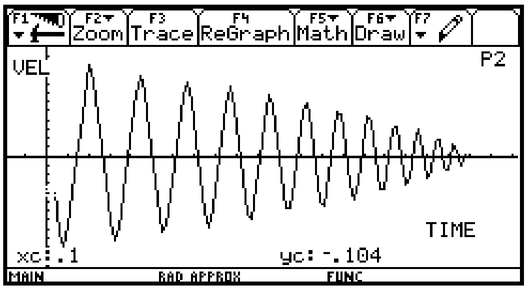
$$\mathbf{a_{ang} = M/I \approx (6g/\ L^2) = constant}$$

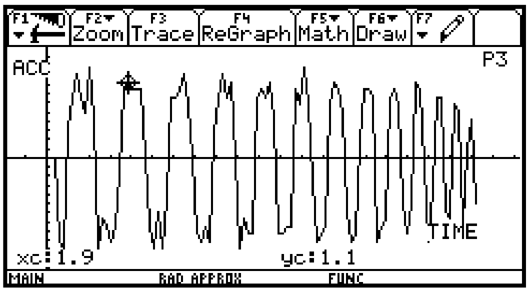
The vertical component of the acceleration of the bar ends is

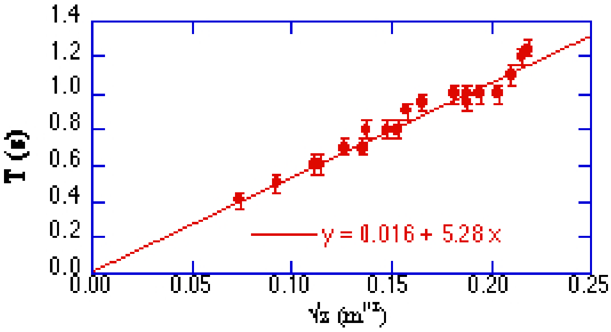
$$\mathbf{a \approx a_{ang}\ (L/2) \approx 3\ g\ d/L}$$

The center of mass “rebounces” *like in the Galileo Oscillator.*





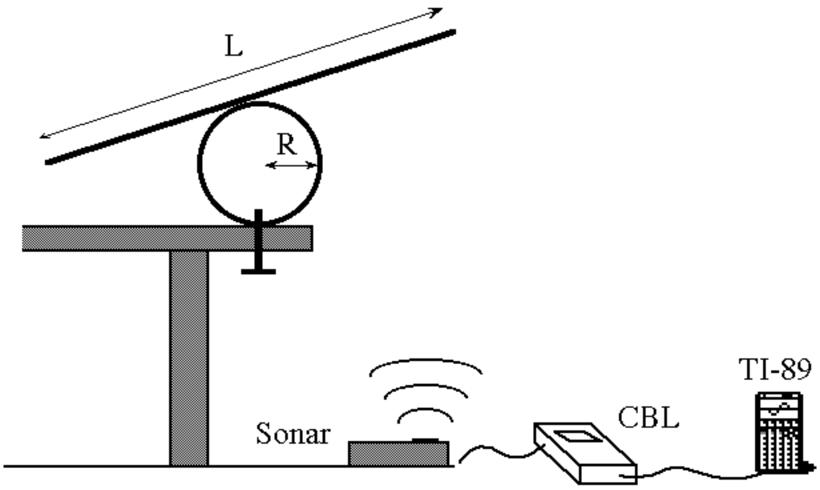




anharmonicity

1. The plots x(t), v(t), a(t) vs. time show the same behavior as in the Galileo Oscillator.
2. Using the small angle approximation the predicted acceleration of the bar end is constant a=3gd/L
3. The period T may be measured and plotted versus the square root of the oscillation amplitude z showing the agreement with the predicted slope $T \approx 4\sqrt{2zL/3gd}$
4. By repeating the experiment with heavy masses at the bar ends (well simulating the motion of a real see-saw) it can be shown that acceleration changes into a=gd/L, 1/3 of the value measured for the unloaded bar.

See-Saw with round pivot



The restoring torque is due to gravity and to pivot reaction.
Torque depends on the oscillation amplitude φ

The effective component of the gravity force is *not constant*:

$$\mathbf{F = m\ g\ cos\ \phi}$$

Here also the lever arm of the torque is *not constant*:

$$\mathbf{b=R\phi}$$

$$\mathbf{M = R\phi\ m\ g\ cos\ \phi}$$

While the momentum of inertia *is nearly constant*:

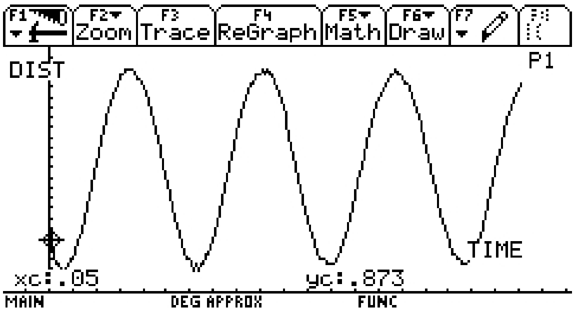
$$\mathbf{I \approx mL^2/12}$$

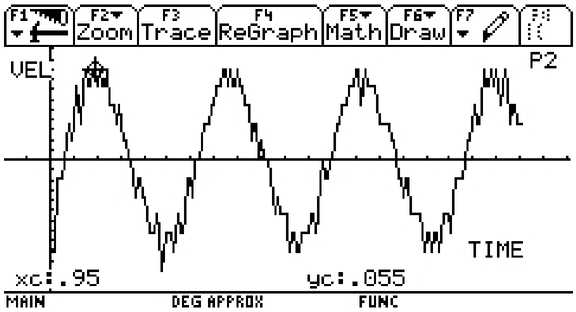
For small oscillations and d<<L

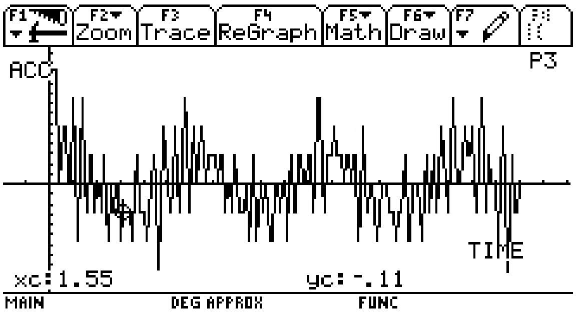
$$\mathbf{a_{ang} = M/I \approx (12Rg/ L^2)\ \phi}$$

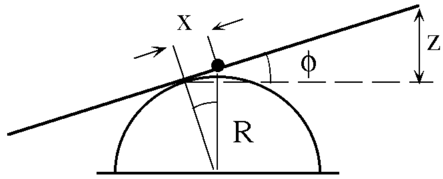
This equation is the same as the equation of a *pendulum* (for small oscillations) with *reduced length* λ**=L²/12R**.

In the same approximation the motion is *harmonic*.
(The same analysis may be applied to the motion of a rocking-chair)





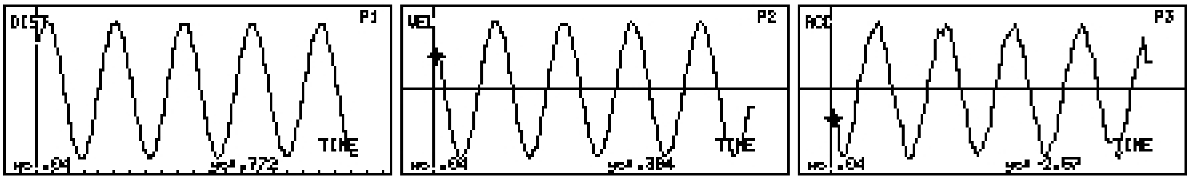




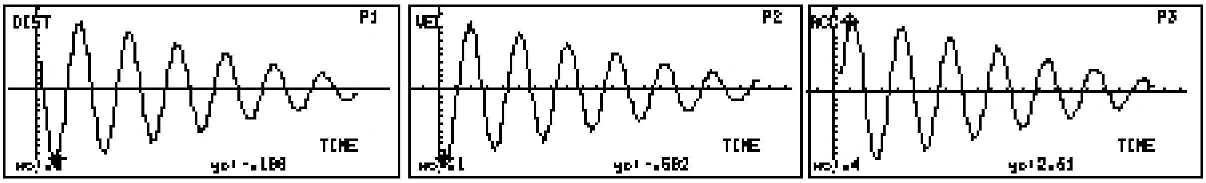
1. The plots x(t), v(t), a(t) vs. time show a different behavior from that of the see-saw with double pivot.
2. Now the period is independent of the amplitude: the motion here is harmonic.
3. The teacher may guide the students to calculate the torque and the moment of inertia, suggesting suitable approximations in order to find the theoretical relationship between angular acceleration α and tilt angle φ: $\alpha \approx -12gR/L^2\varphi$: that is the classic pendumum equation.
4. Also here the oscillation measured with a bar loaded at the ends shows an angular acceleration $\alpha \approx -4gR/L^2\varphi$, reduced of a factor 1/3 with respect to the unloaded bar.

Oscillators comparison

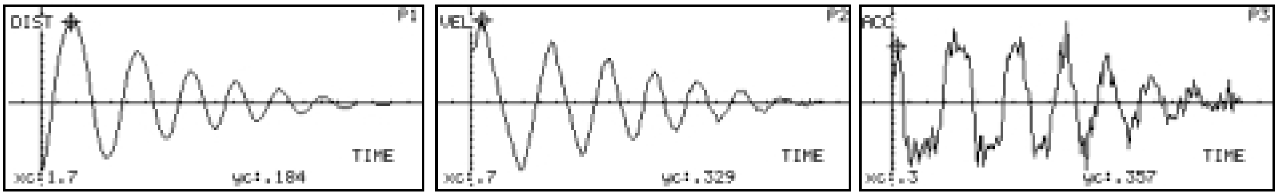
x(t) v(t) a(t)



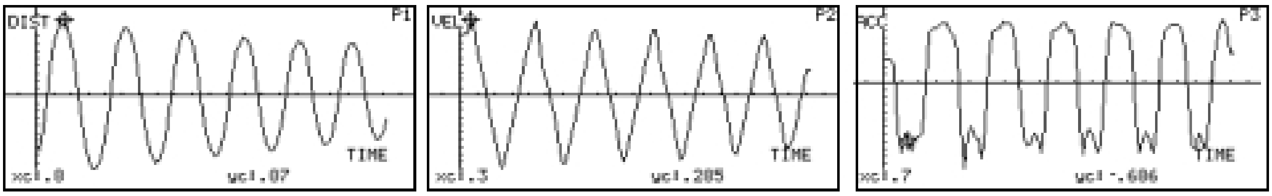
Mass-Spring



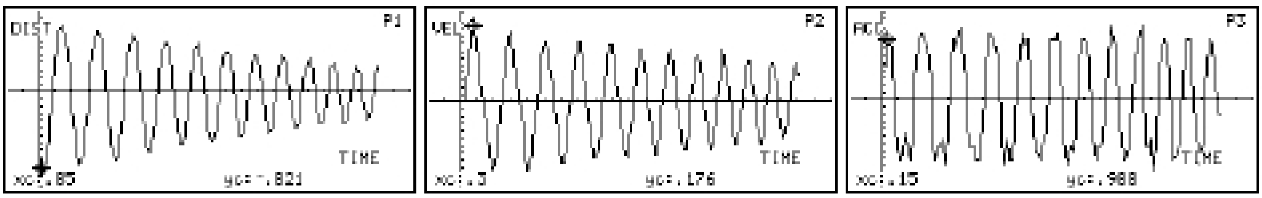
Pendulum Small oscillations



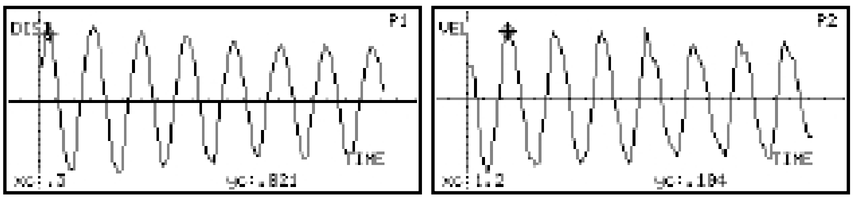
Atwood Oscillator (doble regime)



Galileo Oscillator



See-Saw with double pivot



See-Saw with round pivot